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LETTER TO THE EDITOR

Critical dynamics of the four-state Potts model

S Jain†

Department of Theoretical Physics, Oxford University, Oxford OX1 3NP, UK

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Abstract. The critical relaxation of the four-state Potts model on $L\times L$ lattices with $L\leq 64$ is studied by the Monte Carlo method. The decay of the magnetisation from a non-equilibrium initial state exhibits a crossover from a power-law relaxation for $t \ll t^c(L)$ to an exponential one for $t \gg t^c(L)$, where $t^c(L)$ is some crossover time. This crossover is discussed within the context of dynamic finite-size scaling theory. As a result, the dynamic critical exponent is estimated to be $Z=2.94\pm0.29$.

Although an exact solution of the ferromagnetic two-dimensional q-state Potts model is available for only $q=2$ (the Ising case), the static critical behaviour for $q>2$ is well established [1]. The transition, which is second order for $q\leq 4$, is expected to be first order for $q>4$ [2]. Further, the theoretical predictions for the transition temperature and the static exponents have been confirmed by various techniques [1].

The dynamics, however, are not so well understood. Whereas there are many consistent estimates for the dynamic exponent Z for small q, the latest being [3] $Z(q=2)=2.14\pm0.02$ and [4] $Z(q=3)=2.19\pm0.05$, little is known about $q=4$. The analytic predictions for $Z(q=4)$ can, at best, be viewed as conjectures. By mapping the dynamics of the q-state Potts model on the honeycomb lattice onto the statics of a similar model on the hexagonal close-packed lattice, Domany [5] has suggested that $Z(q=4)=4$. Different suggestions have been made from time-dependent real space renormalisation group studies [6, 7]. These values suffer from the shortcoming that the methods used to obtain them do not reproduce the correct static behaviour and, further, there is also an undetermined error in the actual dynamic exponent quoted.

In this letter we discuss the results of computer simulations of the two-dimensional four-state Potts model at its transition temperature, T_c. Using dynamic finite-size scaling theory [8], we shall obtain the first Monte Carlo estimate for $Z(q=4)$.

The Hamiltonian for the model under investigation can be written as [9]

$$\mathcal{H}=-\sum_{\langle ij\rangle}\delta(\alpha_i,\alpha_j) \qquad (1)$$

where α_i ($\alpha_i=1,2,3,4$) denotes the Potts spin at site i and the summation is over nearest neighbours only. As Boltzmann's constant and the nearest-neighbour interaction are both set to unity, the system has a second-order phase transition at [2, 9] $T_c=[\ln(3)]^{-1}$. We work exactly at T_c with periodic boundary conditions. During the simulations, which are performed on $L\times L$ ($L=4, 8, 16, 32, 64$) lattices, the spins are updated according to the conventional Metropolis algorithm [10]. As we are working

† Present address: Department of Theoretical Chemistry, University Chemical Laboratory, University of Cambridge, Lensfield Road, Cambridge CB2 1EW, UK.

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at a finite temperature, from universality one expects all updating schemes which satisfy detailed balance to yield the same dynamics. This is, of course, not necessarily true if the system in question has a zero temperature transition [11].

In this work we concentrate on the decay of the magnetisation, $M(t)$, given by

$$M(t) = \tfrac{4}{3}\left(L^{-2} \sum_i \delta(1, \alpha_i(t)) - \tfrac{1}{4} \right) \tag{2}$$

where $\alpha_i(t)$ indicates the value of the ith spin at time t. For each L, we start with $M(t=0)=1$ and record the subsequent behaviour of the magnetisation. To obtain reliable statistics one has to average over many samples; see table 1 for details—the contents of the final two rows are discussed later.

From dynamic finite-size scaling [8], the magnetisation of a system of size $L \times L$ at T_c in zero field evolves according to

$$M(t, L) = L^{-\beta/\nu} f(tL^{-Z}) \tag{3}$$

where β and ν are static exponents and f is a scaling function which has the following limiting form:

$$f(X) = \begin{cases} AX^{-\beta/\nu Z} & \text{as } X \to 0 \quad (4a) \\ B\exp(-CX) & \text{as } X \to \infty \quad (4b) \end{cases}$$

and here A, B and C are constants. Thus, there is a crossover time $t^c(L)$ such that for $t \ll t^c(L)$ one obtains equation $(4a)$, while for $t \gg t^c(L)$ exponential decay is expected. Further, $t^c(L)$ is a monotonically increasing function of L.

In figure 1 we show a log–log plot of the magnetisation against t for $L \leqslant 64$. Note that where the data for $L < 64$ are indistinguishable from those for $L = 64$ (see table 1), only the largest size is displayed. For each $L < 64$, there is a crossover from power-law decay at short times to a faster one at longer times. To see if the faster decay is indeed an exponential one, we re-plot the data for $t \to \infty$ as $\ln M(t)$ against

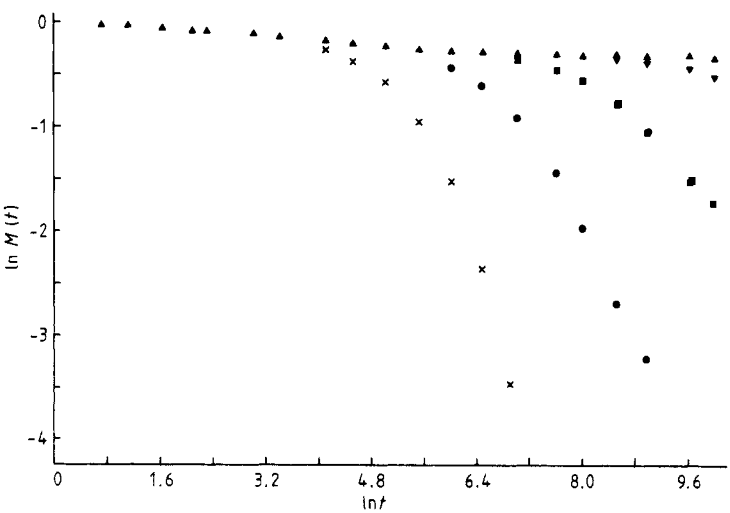

Figure 1. A plot of $\ln M(t)$ against $\ln t$ for $L \leqslant 64$. Where the data overlap (in practice, if they are within 2% of each other), only those for $L = 64$ are shown. ▲, 64×64; ▼, 32×32; ■, 16×16; ● 8×8; ×, 4×4.

Table 1. Details of the simulations.

System size, L	4	8	16	32	64
Number of samples	33 792	8320	2176	576	124
Range of t over which the data overlap with $L=64$	0–30	0–250	0–650	0–3000	
$\tau(L)$	292 ± 4	1628 ± 54	9911 ± 321	$103\,506 \pm 2987$	—

t. As can be seen from figures 2(a)–(d), the finite-size effects are consistent with equation (4b).

From equations (3) and (4b) we see that one can define a characteristic decay time, $\tau(L)$, by

$$\tau(L) = C^{-1}L^{Z} \tag{5}$$

thereby enabling one to evaluate the dynamic critical exponent. Figure 3 shows a plot of $\ln \tau(L)$ against $\ln L$, where the values of $\tau(L)$ (see table 1) have been obtained from the gradients of the best fits shown in figures 2(a)–(d). The above relationship (equation (5)) is expected to hold for $L \gg 1$, i.e. for small L one expects corrections to finite-size scaling. In the present case, however, $L=4$ does not appear to show any appreciable deviation from the other sizes considered. Fitting all of the data to equation (5) gives $Z \sim 2.75$, whereas a restricted fit over the last three points yields $Z \sim 3.01$. As there is some uncertainty involved in extracting $\tau(L)$, we should like to quote an average value of $Z = 2.94 \pm 0.29$ as our final estimate. Table 2 gives our result together with the other predictions which have been made [5–7].

As mentioned above, for $t \ll t^{c}(L)$ one expects bulk-like power-law decay. We test for this behaviour in figure 4 which shows a magnified plot of figure 1 over the range $2 \leqslant t \leqslant 3000$. In order to improve the statistics, all of the available data—where they overlap—have been averaged. From equations (3), (4a) and the line of best fit over $10 \leqslant t \leqslant 3000$, which is shown in figure 4, we get $\beta/\nu Z = 4.595 \times 10^{-2} \mp 1.94 \times 10^{-3}$. Combining this result with our estimate for Z leads to $\beta/\nu = 0.135 \pm 0.019$; this compares favourably with the exact value of 0.125 [1].

Finally, as a self-consistent check, in figure 5 we plot $\ln[L^{\beta/\nu}M(t, L)]$ against $\ln[tL^{-Z}]$ with $\beta/\nu = 0.125$ and $Z = 2.94$. This plot clearly shows that all of the data—including those for $L=4$—can be explained by dynamic finite-size scaling. Note that such a plot becomes progressively worse as Z is increased or decreased; in particular, we can exclude [5] $Z=4$ on the basis of these data.

At this point it is worth noting that there are systematic errors involved in the present analysis. Strictly speaking, the dynamic exponent quoted above depends both on L and $M(t^*, L)$, where t^* is the longest time simulated for any given L, that is

$$Z = \lim_{L\to\infty} \lim_{t^*\to\infty} Z(t^*, L). \tag{6}$$

In the present work, $M(t^*, L)$ varies considerably: $M(t^* = 1000, 4) \sim 0.03$ and $M(t^* = 22\,500, 32) \sim 0.59$. Consequently, for any given L, one introduces an error in the corresponding value of $\tau(L)$. It should be made clear that the error bars quoted for τ in table 1 do not take into account any systematic errors. As mentioned above, there is a further error—this time due to corrections to finite-size scaling—involved in

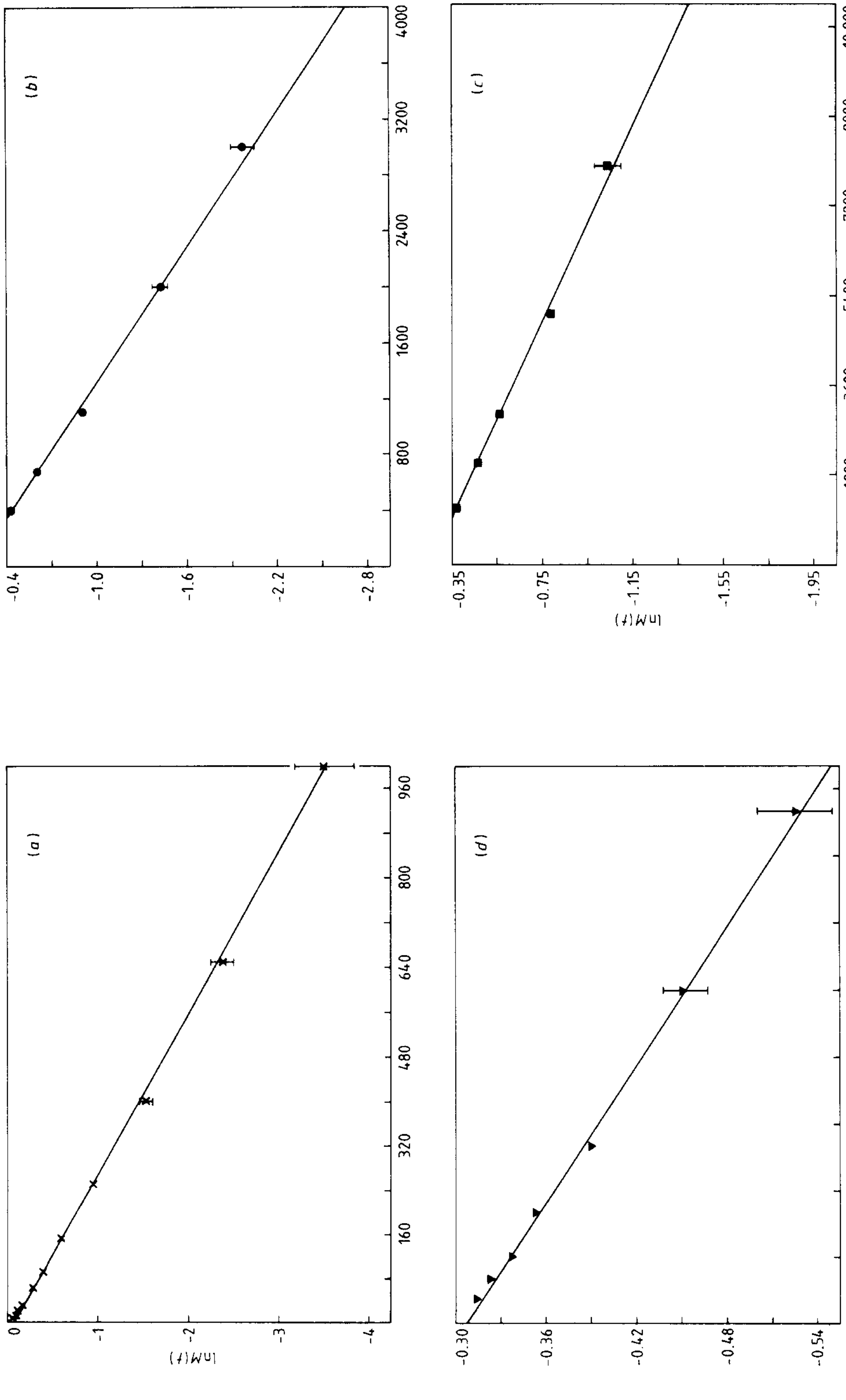

Figure 2. ln $M(t)$ against t for L = (a) 4, (b) 8, (c) 16 and (d) 32. The values of $\tau(L)$ resulting from the various best fits ($L = 4$: $60 \leq t \leq 1000$; $L = 8$: $400 \leq t \leq 3000$; $L = 16$: $1100 \leq t \leq 8000$; $L = 32$: $5000 \leq t \leq 22\,500$) are given in table 1.

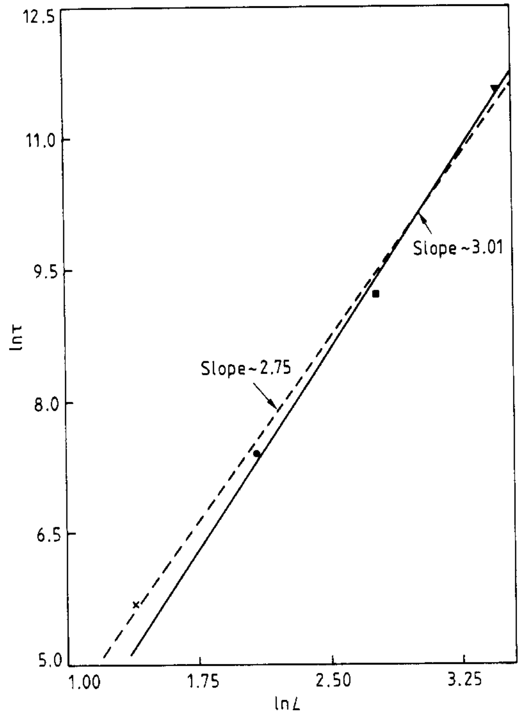

Figure 3. A log-log plot of τ against L. The best fits for $L = 8$, 16, 32 (full line) and for $4 \leq L \leq 32$ (broken line) are also shown.

Table 2. The critical dynamic exponent Z for the two-dimensional four-state Potts model.

Reference	Value of Z
Domany [5]	4
Forgacs *et al* [6]	2.55
Lage [7]	2.68
This work	2.94 ± 0.29

extracting Z from $\tau(L)$. As can be seen from figure 5, there do not appear to be either any significant finite-size nor systematic corrections. So, we can be fairly confident of our final estimate for the dynamic critical exponent.

To conclude, we have used dynamic finite-size scaling to investigate the decay of the magnetisation from an ordered state. We do not find any appreciable corrections to finite-size scaling for the sizes considered. Our final estimate of $Z(q=4)$, which certainly excludes the Domany [5] conjecture, indicates that the four-state Potts model probably does not belong to the same dynamic universality class as $q = 2$ or 3.

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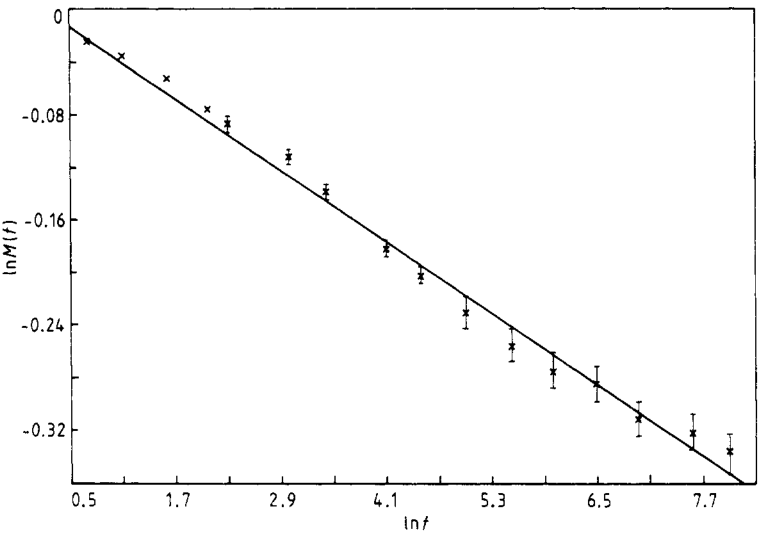

Figure 4. A magnified plot of some of the data shown in figure 1. Only the data for $t \leq 3000$ are shown (averaged over as many sizes as possible) and the line of best fit is for $10 \leq t \leq 3000$. This has a gradient $(= -\beta/\nu Z)$ of $-4.595 \times 10^{-2} \pm 1.94 \times 10^{-3}$.

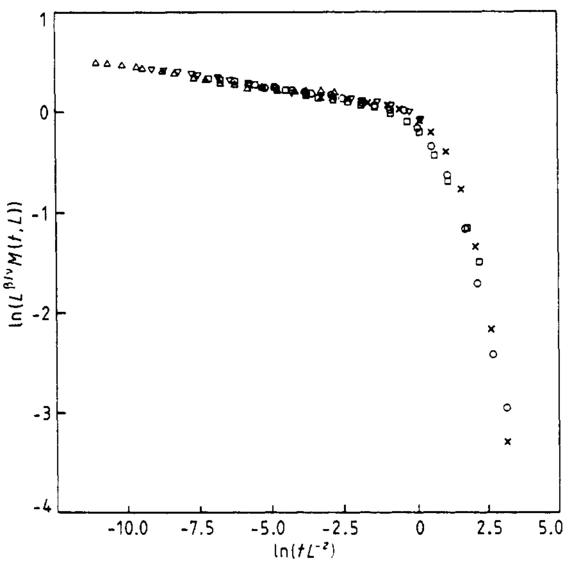

Figure 5. A finite-size scaling plot of the data with $\beta/\nu = 0.125$ and $Z = 2.94$. △, 64×64; ▽, 32×32; □, 16×16; ○, 8×8; ×, 4×4.

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